Master theorem

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Bachman-Landau notation and the Master theorem for divide-and-conquer recurrences

This project's goal is to formalize a proof of the Master theorem for divide-and-conquer recurrences. Along with the Master theorem, we also state definitions and some properties of the Big-O and its sibling notations.

We use original and standard definitions of the Big-O notation [1, 4, 5], as well as Knuth's definition of Big-Omega and Big-Theta [3]. We use a slightly adapted formulation of the Master Theorem from Cormen et al. [2], with the difference being due to our use of functions on natural numbers as opposed to their more general formulation. The idea of the formalized proof is also adapted from the same source, but their proof skips many technical steps, so our proof appears to hold little resemblance.

Chapter 1

Basic asymptotic properties of functions

Definition 1. f is asymptotically positive if there exists x_0 such that $f(x) > 0$ for all $x \geq x_0$.

Definition 2. f is asymptotically negative if there exists x_0 such that $f(x) < 0$ for all $x \geq x_0$.

Definition 3. f is asymptotically nonpositive if there exists x_0 such that $f(x) \geq 0$ for all $x \geq x_0$.

Definition 4. f is asymptotically negative if there exists x_0 such that $f(x) \leq 0$ for all $x \geq x_0$.

Definition 5. f is asymptotically less than g if there exists x_0 such that $f(x) \leq g(x)$ for all $x \geq x_0$.

Definition 6. f is asymptotically greater than g if there exists x_0 such that $f(x) \geq g(x)$ for all $x \geq x_0$.

1.1 Asymptotic positivity and negativity

Lemma 7. *If f is asymptotically negative, then $-f$ is asymptotically positive.*

Proof. By definition of asymptotic positivity, there exists an x_0 such that $f(x) > 0$ for all $x > x\_0$. It follows that $-f(x) > 0$, which is what is needed. ☐

Lemma 8. *If f is asymptotically positive, then $-f$ is asymptotically negative.*

Proof. By definition of asymptotic negativity, there exists an x_0 such that $f(x) < 0$ for all $x > x\_0$. It follows that $-f(x) < 0$, which is what is needed. ☐

1.2 Asymptotic inequality

1.2.1 Positivity and negativity

Lemma 9. *Let f be asymptotically positive and let it be asymptotically less than g. Then g is asymptotically positive.*

Proof. Assume with no loss of generality that $f(x) > 0$ for all $x > x\_0$ and that $f(y) \leq g(y)$ for all $y > y\_0$. Let $z\_0 = \max\{x\_0, y\_0\}$. By transitivity of the inequality relations, we have $g(z) > 0$ for all $z > z\_0$. □

Lemma 10. *Let f be asymptotically less than g and let g be asymptotically negative. Then f is asymptotically negative.*

Proof. Assume with no loss of generality that $g(x) < 0$ for all $x > x\_0$ and that $f(y) \leq g(y)$ for all $y > y\_0$. Let $z\_0 = \max\{x\_0, y\_0\}$. By transitivity of the inequality relations, we have $f(z) < 0$ for all $z > z\_0$. □

Lemma 11. *Let f be asymptotically greater than g and let g be asymptotically negative. Then g is asymptotically negative.*

Proof. By Lemma 17, this statement is equivalent to Lemma 9. □

Lemma 12. *Let f be asymptotically negative and let it be asymptotically greater than g. Then g is asymptotically negative.*

Proof. By Theorem 17, this statement is equivalent to Lemma 10. □

1.2.2 Reflexivity

Lemma 13. *f is asymptotically less than f.*

Proof. By reflexivity of $\leq$, we have $f(x) \leq f(x)$ for any given x. □

Lemma 14. *f is asymptotically greater than f.*

Proof. By reflexivity of $\geq$, we have $f(x) \geq f(x)$ for any given x. □

1.2.3 Equivalence

Lemma 15. *Let f be asymptotically less than g. Then g is asymptotically greater than f.*

Proof. Since $f(x) \leq g(x)$ for all $x > x\_0$, we have $g(x) \geq f(x)$. □

Lemma 16. *Let f be asymptotically greater than g. Then g is asymptotically less than f.*

Proof. Since $f(x) \geq g(x)$ for all $x > x\_0$, we have $g(x) \leq f(x)$. □

Theorem 17. *f is asymptotically less than g if and only if g is asymptotically greater than f.*

Proof. Lemma 15 and Lemma 16 are both directions respectively. □

1.2.4 Transitivity

Lemma 18. *If f is asymptotically less than g and g is asymptotically less than h, then f is asymptotically less than h.*

Proof. By assumption, $f(x) \leq g(x)$ for all $x \geq x\_0$ and $g(y) \leq h(y)$ for all $y \geq y\_0$. Let $z\_0 = \max\{x\_0, y\_0\}$. By transitivity, we have $f(z) \leq g(z)$ for all $z \geq z\_0$. □

Lemma 19. *If f is asymptotically greater than g and g is asymptotically greater than h, then f is asymptotically greater than h.*

Proof. By the equivalence given by Lemma 17, we can apply Lemma 18 in reverse, since h is asymptotically less than g and g is asymptotically less than f. □

1.2.5 Additivity

Lemma 20. *Let f_1 be asymptotically less than g_1 and f_2 be asymptotically less than g_2. Then $f\_1 + f\_2$ is asymptotically less than $g\_1 + g\_2$.*

Proof. Let x_0 be such that $f\_1(x) \leq g\_1(x)$ for all $x > x\_0$ and let y_0 be such that $f\_2(y) \leq g\_2(y)$ for all $y > y\_0$. Those exist due to assumptions. Now let $z\_0 = \max\{x\_0, y\_0\}$. By transitivity, $f\_1(z) \leq g\_1(z)$ and $f\_2(z) \leq g\_2(z)$ for all $z > z\_0$. By additivity, we can merge both inequalities by adding both terms on the left side and both terms on the right side. We thus get $f\_1(z) + f\_2(z) \leq g\_1(z) + g\_2(z)$, which by definition and and extensionality means that $f\_1 + f\_2$ is asymptotically less than $g\_1 + g\_2$. □

Lemma 21. *Let f_1 be asymptotically greater than g_1 and f_2 be asymptotically greater than g_2. Then $f\_1 + f\_2$ is asymptotically greater than $g\_1 + g\_2$.*

Proof. By Theorem 17, g_1 and g_2 are asymptotically less than f_1 and f_2 respectively. It suffices to show that $g\_1 + g\_2$ is asymptotically less than $f\_1 + f\_2$, which is precisely the result of Lemma 20. □

Lemma 22. *Let f_1 be asymptotically positive. Let also f_2 be asymptotically greater than g. Then $f\_1 + f\_2$ is asymptotically greater than g.*

Proof. By definition, there exists some x_0 such that $f\_1(x) > 0$ for all $x > x\_0$. We also have $f\_2(y) \geq g(y)$ for all $y > y\_0$ for some y_0. Let $z\_0 = \max\{x\_0, y\_0, \}$. We now have, for all $z > z\_0$ both $f\_1(z) > 0$ and $f\_2(z) \geq g(z)$. By additivity, we have $f\_1(z) + f\_2(z) \geq g(z)$. □

Lemma 23. *Let f_1 be asymptotically negative. Let also f_2 be asymptotically less than g. Then $f\_1 + f\_2$ is asymptotically less than g.*

Proof. By definition, there exists some x_0 such that $f\_1(x) < 0$ for all $x > x\_0$. We also have $f\_2(y) \leq g(y)$ for all $y > y\_0$ for some y_0. Let $z\_0 = \max\{x\_0, y\_0, \}$. We now have, for all $z > z\_0$ both $f\_1(z) < 0$ and $f\_2(z) \leq g(z)$. By additivity, we have $f\_1(z) + f\_2(z) \leq g(z)$. □

1.2.6 Scalar multiplicativity

Lemma 24. *Let $c > 0$ and let f be asymptotically less than g. Then $c \cdot f$ is asymptotically less than $c \cdot g$.*

Proof. This is a simple consequence of scalar multiplication by a positive constant. □

Lemma 25. *Let $c > 0$ and let f be asymptotically greater than g. Then $c \cdot f$ is asymptotically greater than $c \cdot g$.*

Proof. By applying Theorem 17, the proof boils down to proving that $c \cdot g$ is asymptotically less than $c \cdot f$, which is precisely shown by Lemma 24. □

Lemma 26. *Let $c < 0$ and let f be asymptotically less than g. Then $c \cdot f$ is asymptotically greater than $c \cdot g$.*

Proof. This is a simple consequence of the fact that if $f(x) \leq g(x)$, then for a $c < 0$ we have $c \cdot f(x) \geq c \cdot g(x)$. □

Lemma 27. *Let $c < 0$ and let f be asymptotically greater than g. Then $c \cdot f$ is asymptotically less than $c \cdot g$.*

Proof. Similar to above, the proof is a direct application of Theorem 17 and Lemma 26. □

1.2.7 Multiplicativity

Theorem 28. *Let f_1 and f_2 be asymptotically nonnegative functions. If f_1 is asymptotically less than g_1 and f_2 is asymptotically less than g_2, then $f\_1 * f\_2$ is asymptotically less than $g\_1 * g\_2$.*

Proof. Asymptotic properties give constants $x\_i, 0 \leq i \leq 3$, above which each property holds. We take $x_M = \max_{0 \leq i \leq 3} x_i$ as the constant of the needed property. For all $x > x_M$, all given asymptotic properties hold, so the wanted property holds by properties of the inequality relation. □

Theorem 29. *Let f_1 and f_2 be asymptotically nonpositive functions. If f_1 is asymptotically greater than g_1 and f_2 is asymptotically greater than g_2, then $f\_1 * f\_2$ is asymptotically less than $g\_1 * g\_2$.*

Proof. Analogously to above, the proof comes from taking the maximum of asymptotic constants as the asymptotic lower bound for nonpositivity. This time, however, the inequality flips due to nonpositive terms in $f\_1(n) * f\_2(n) \leq g\_1(n) * g\_2(n)$, since n is larger than all of the asymptotic lower bounds. □

Chapter 2

Asymptotic growth

Definition 30. f is asymptotically bounded above by g if there exists a $k > 0$ such that f is asymptotically less than $k * g$.

Definition 31. f is asymptotically bounded below by g if there exists $k > 0$ such that f is asymptotically greater than $k * g$.

Definition 32. f is asymptotically bounded by g if f is asymptotically bounded above and below by g.

Definition 33. f is asymptotically dominated by g if for all $k > 0$ f is asymptotically less than $k * g$.

Definition 34. f asymptotically dominates g if for all $k > 0$ (x) is asymptotically greater than $k * g$.

Lemma 35. *If f is dominated by g, then it's also bounded above by g.*

Proof. The definitions of f being dominated and bounded above by g only differ in the quantifier before k at the very start (universal for the hypothesis, existential for the goal), so it suffices to use any positive value for k. We can use 1. The desired result then follows directly. □

Lemma 36. *If f dominates g, then it's bounded below by g.*

Proof. The proof is entirely analogous to the previous proof. □

Lemma 37. *f is asymptotically bounded above and below by g if and only if f is asymptotically bounded by g.*

Proof. Both directions follow directly from the definition of asymptotic boundedness. □

Lemma 38. *Let g be asymptotically positive. Then f is not both asymptotically bounded below by g and asymptotically dominated by g.*

Proof. Suppose f is asymptotically bounded below by g and also asymptotically dominated by g. We need to find a contradiction to prove the statement.

First, we claim that there exists x_0 such that for all $x \geq x\_0$ we have $f(x) \geq k \cdot g(x)$ for some $k > 0$, $g(x) \geq 0$ and $f(x) \leq k' \cdot g(x)$ for all $k' > 0$. Each of the asymptotic assumption gives one such constant, so taking the maximum of all three gives the needed value.

For the contradiction, consider $f(x) \leq k' \cdot g(x)$ when $k' = k/2$. In this case, we have $f(x) \leq (k/2) \cdot g(x)$, but we also have $k \cdot g(x) \leq f(x)$, leading by transitivity to $k \cdot g(x) \leq (k/2) \cdot g(x)$, an obvious contradiction to the fact that $(k/2) \cdot g(x) \leq k \cdot g(x)$. □

Theorem 39. *Let f be asymptotically positive. If f is asymptotically bounded below by g, then f is not asymptotically dominated by g.*

Proof. This is a direct application of Lemma 38. $\square$

Theorem 40. *Let f be asymptotically positive. If f is asymptotically bounded below by g, then f is not asymptotically dominated by g.*

Proof. This is a direct application of Lemma 38. $\square$

Lemma 41. *Let g be asymptotically positive. Then it does not hold that both f is asymptotically bounded above by g and f asymptotically dominate g.*

Proof. The proof is analogous to the proof of Lemma 38. This time, we set $k' = k + 1$ and thus produce the false inequality $(k + 1) \cdot g(x) \leq k \cdot g(x)$. $\square$

Theorem 42. *Let f be asymptotically positive. If f is bounded below by g, then f does not dominate g.*

Proof. This is a direct application of Lemma 41. $\square$

Theorem 43. *Let f be asymptotically positive. If f is asymptotically bounded above by g, then f does not asymptotically dominate g.*

Proof. This is a direct application of Lemma 41. $\square$

Lemma 44. *Let g be asymptotically positive. Then it is not true that both f is asymptotically dominated by g and that f dominates g.*

Proof. Suppose f both dominates g and is dominated by g. Our goal is to find a contradiction. We have by definition the inequalities $g(x) > 0$, $f(x) \geq k\_1 \cdot g(x)$ and $f(x) \leq k\_2 \cdot g(x)$ for all $k\_1 > 0$, $k\_2 > 0$ and for all $x \geq x\_0$ for some x_0. Note that we use the same constant x_0 for all inequalities with no loss of generality. In fact, we shall also use the same x_0 in the asymptotic positivity condition $g(x) > 0$.

Fix $k\_1 = 2$ and $k\_2 = 1$ (generally, we only need $k\_1 \geq k\_2$), so we now have $f(x) \geq 2 \cdot g(x)$ and $f(x) \leq g(x)$. From these inequalities, it immediately follows that $2 \cdot g(x) \leq g(x)$. However, since $1 \leq 2$ and $g(x) > 0$, we have $g(x) < 2 \cdot g(x)$. We thus have two contradicting inequalities, finishing the proof. $\square$

Theorem 45. *Let g be asymptotically positive. If f is asymptotically dominated by g, then f does not asymptotically dominate g.*

Proof. This is a direct application of Lemma 44. $\square$

Theorem 46. *Let g be asymptotically positive. If f asymptotically dominates g, then f is not asymptotically dominated by g.*

Proof. This is a direct application of Lemma 44. $\square$

2.1 Reflexivity

Lemma 47. *f is asymptotically bounded by itself.*

Proof. Proving asymptotic boundedness is equivalent to proving boundedness above and below. Both can be proved the same way - we choose 1_K for K and 1_R for N, then the required asymptotic growth properties follow from definitions of identity elements for K and R. □

Lemma 48. *f is asymptotically bounded above by itself.*

Proof. This follows directly from 47. □

Lemma 49. *f is asymptotically bounded below by itself.*

Proof. This follows directly from 47. □

2.2 Transitivity

Lemma 50. *If f is asymptotically bounded above by g and g is asymptotically bounded above by h, then f is asymptotically bounded above by h.*

Proof. Let k_1 and k_2 be the constants such that $f(x) \leq k\_1 \cdot g(x)$ and $g(x) \leq k\_2 \cdot h(x)$ for sufficiently large x. By multiplicativity and transitivity, we have $f(x) \leq k\_1 \cdot k\_2 \cdot h(x)$. □

Lemma 51. *If f is asymptotically bounded below by g and g is asymptotically bounded below by h, then f is asymptotically bounded below by h.*

Proof. Let k_1 and k_2 be the constants such that $f(x) \geq k\_1 \cdot g(x)$ and $g(x) \geq k\_2 \cdot h(x)$ for sufficiently large x. By multiplicativity and transitivity, we have $f(x) \geq k\_1 \cdot k\_2 \cdot h(x)$. □

Lemma 52. *If f is asymptotically bounded by g and g is asymptotically bounded by h, then f is asymptotically bounded by h.*

Proof. A direct consequence of Lemma 50 and Lemma 51. □

Lemma 53. *If f is asymptotically dominated by g and g is asymptotically dominated by h, then f is asymptotically dominated by h.*

Proof. Let $k > 0$. Then by first assumption, we have $f(x) \leq k \cdot g(x)$ for large x. By the second assumption, we have $g(x) \leq h(x)$ for large x. By multiplicativity and transitivity, we get $f(x) \leq k \cdot h(x)$. □

Lemma 54. *If f asymptotically dominates g and g asymptotically dominates h, then f asymptotically dominates h.*

Proof. Let $k > 0$. Then by first assumption, we have $f(x) \geq k \cdot g(x)$ for large x. By the second assumption, we have $g(x) \geq h(x)$ for large x. By multiplicativity and transitivity, we get $f(x) \geq k \cdot h(x)$. □

2.3 Scalar multiplicativity

Lemma 55. *Let $c > 0$. If f is bounded above by g, then $c \cdot f$ is also bounded above by g.*

Proof. Let k be the constant such that f is asymptotically less than $k \cdot g$. By positive scalar multiplicativity of asymptotic inequality, $c \cdot f$ is asymptotically less than $c \cdot k \cdot g$. Since $c \cdot k > 0$, this implies that $c \cdot f$ is bounded above by g. □

Lemma 56. *Let $c > 0$. If f is bounded below by g, then $c \cdot f$ is also bounded below by g.*

Proof. Let k be the constant such that f is asymptotically greater than $k \cdot g$. By positive scalar multiplicativity of asymptotic inequality, $c \cdot f$ is asymptotically greater than $c \cdot k \cdot g$. Since $c \cdot k > 0$, this implies that $c \cdot f$ is bounded below by g. □

Lemma 57. *Let $c > 0$. If f is bounded by g, then $c \cdot f$ is also bounded by g.*

Proof. Above boundedness is exactly Lemma 55 and below boundedness is exactly Lemma 56. □

Lemma 58. *Let $c < 0$. If f is bounded above by g, then $c \cdot f$ is bounded above by $-g$.*

Proof. Let k be the constant such that f is asymptotically less than $k \cdot g$. By positive scalar multiplicativity of asymptotic inequality, $-c \cdot f$ is asymptotically less than $-c \cdot k \cdot g$. Since $-c \cdot k > 0$, this implies that $c \cdot f$ is bounded above by $-g$. □

Lemma 59. *Let $c < 0$. If f is bounded below by g, then $c \cdot f$ is bounded below by $-g$.*

Proof. Let k be the constant such that f is asymptotically greater than $k \cdot g$. By positive scalar multiplicativity of asymptotic inequality, $-c \cdot f$ is asymptotically greater than $-c \cdot k \cdot g$. Since $-c \cdot k > 0$, this implies that $c \cdot f$ is bounded below by $-g$. □

Lemma 60. *Let $c < 0$. If f is bounded by g, then $c \cdot f$ is bounded by $-g$.*

Proof. Above boundedness is exactly Lemma 58 and below boundedness is exactly Lemma 59. □

2.4 Additivity

Lemma 61. *Let f_1 and f_2 be bounded above by g. Then $f\_1 + f\_2$ is also bounded above by g.*

Proof. Let k_1 and k_2 be the constants such that f_1 is asymptotically less than $k\_1 \cdot g$ and f_2 is asymptotically less than $k\_2 \cdot g$. By additivity of asymptotic inequality, $f\_1 + f\_2$ is asymptotically less than $(k\_1 * k\_2) \cdot g$. It directly follows that $f\_1 + f\_2$ is asymptotically bounded above by g. □

Lemma 62. *Let f_1 and f_2 be bounded below by g. Then $f\_1 + f\_2$ is also bounded below by g.*

Proof. Let k_1 and k_2 be the constants such that f_1 is asymptotically greater than $k\_1 \cdot g$ and f_2 is asymptotically greater than $k\_2 \cdot g$. By additivity of asymptotic inequality, $f\_1 + f\_2$ is asymptotically greater than $(k\_1 * k\_2) \cdot g$. It directly follows that $f\_1 + f\_2$ is asymptotically bounded below by g. □

Lemma 63. *Let f_1 and f_2 be bounded by g. Then $f\_1 + f\_2$ is also bounded by g.*

Proof. This is proved directly by Lemma 61 and Lemma 62. □

Lemma 64. *Let f_1 be bounded below by g and let f_2 be asymptotically positive. Then $f\_1 + f\_2$ is bounded below by g.*

Proof. This property immediately follows from Lemma 22. □

Lemma 65. *Let f_1 be bounded above by g and let f_2 be asymptotically negative. Then $f\_1 + f\_2$ is bounded above by g.*

Proof. This property immediately follows from Lemma 23. □

Lemma 66. *Let f_1 be bounded by g. Let also f_2 be asymptotically positive and bounded above by g. Then $f\_1 + f\_2$ is bounded by g.*

Proof. This property immediately follows from Lemma 61 and Lemma 64. □

Lemma 67. *Let f_1 be bounded by g. Let also f_2 be asymptotically negative and bounded below by g. Then $f\_1 + f\_2$ is bounded by g.*

Proof. This property immediately follows from Lemma 62 and Lemma 65. □

Lemma 68. *Let f_1 be bounded by g. Let also f_2 be asymptotically positive and let f_2 be dominated by g. Then $f\_1 + f\_2$ is bounded by g.*

Proof. We prove this with an application of Lemma 66 on Lemma 35. □

Lemma 69. *Let f_1 be bounded by g. Let also f_2 be asymptotically negative and let f_2 dominate g. Then $f\_1 + f\_2$ is bounded by g.*

Proof. We prove this with an application of Lemma 67 on Lemma 36. □

2.5 Multiplicativity

Lemma 70. *Let f_1 and f_2 be asymptotically nonnegative functions such that f_1 is asymptotically bounded above by g_1 and f_2 is asymptotically bounded above by g_2. Then $f\_1 * f\_2$ is asymptotically bounded above by $g\_1 * g\_2$.*

Proof. Let k_1 and k_2 be the constants such that f_1 is asymptotically less than g_1 and f_2 is asymptotically less than g_2. By multiplicativity of asymptotic inequality, $f\_1 * f\_2$ is asymptotically less than $k\_1 \cdot g\_1 * k\_2 \cdot g\_2$, which is equivalent to asymptotic above boundedness of $f\_1 * f\_2$ by $g\_1 * g\_2$. □

Lemma 71. *Let f_1 and f_2 be asymptotically nonpositive functions such that f_1 is asymptotically bounded below by g_1 and f_2 is asymptotically bounded below by g_2. Then $f\_1 * f\_2$ is asymptotically bounded below by $g\_1 * g\_2$.*

Proof. Let k_1 and k_2 be the constants such that f_1 is asymptotically greaterthan g_1 and f_2 is asymptotically greater than g_2. By multiplicativity of asymptotic inequality, $f\_1 * f\_2$ is asymptotically less than $k\_1 \cdot g\_1 * k\_2 \cdot g\_2$, which is equivalent to asymptotic below boundedness of $f\_1 * f\_2$ by $g\_1 * g\_2$. □

Chapter 3

Bachman-Landau notation

Bachman-Landau family of notations is the name of a few closely related notations used in algorithm analysis. The most famous of them is the so-called big-O notation. While most formulations are defined on functions from naturals or reals to reals, we define them more generally - requirements for the types of the domain and codomain vary between different properties. However, all properties hold for functions from a linearly ordered commutative ring to a linearly ordered field. In the rest of this page, we let R be a linearly ordered commutative ring and F be a linearly ordered field. We will also use symbols f, f__1, f__2, g, g__1 and g__2 for functions $R \to F$. Also, we let M be a right R-module, although often only a (distributive) left multiplicative action on R is required.

3.1 Asymptotic sets

Definition 72. (Big O notation) $f(x) \in O(g(x))$ if it is asymptotically bounded above by $g(x)$.

Definition 73. (Big Omega notation) $f(x) \in \Omega(g(x))$ if it is asymptotically bounded below by $g(x)$.

Definition 74. (Big Theta notation) $f(x) \in \Theta(g(x))$ if it is asymptotically bounded by $g(x)$.

Definition 75. (Small O notation) $f(x) \in o(g(x))$ if it is asymptotically dominated by $g(x)$.

Definition 76. (Small Omega notation) $f(x) \in \omega(g(x))$ if it asymptotically dominates $g(x)$.

3.2 Relations between asymptotic sets

Lemma 77. *If $f(x) \in o(g(x))$, then $f(x) \in O(f(x))$.*

Proof. Since $o(g(x))$ and $O(f(x))$, we can simply use Lemma 35. ☐

Theorem 78. *If $f(x) \in \omega(g(x))$, then $f(x) \in \Omega(g(x))$.*

Proof. The proof is a simple application of Theorem 36. ☐

Theorem 79. *$f(x) \in O(g(x))$ and $f(x) \in \Omega(g(x))$ if and only if $f(x) \in \Theta(g(x))$.*

Proof. Similarly to previous proofs, the proof is a direct application of Lemma 37. ☐

Lemma 80. *Let g be asymptotically positive. Then $f(x) \in \Theta(g(x))$ and $f(x) \in o(g(x))$ are not both true.*

Proof. A direct application of Lemma 38. □

Lemma 81. *Let g be asymptotically positive. If $f(x) \in \Theta(g(x))$ then $f(x) \notin o(g(x))$.*

Proof. A direct application of Lemma 80. □

Lemma 82. *Let g be asymptotically positive. If $f(x) \in o(g(x))$ then $f(x) \notin \Theta(g(x))$.*

Proof. A direct application of Lemma 80. □

Lemma 83. *Let g be asymptotically positive. If $f(x) \in \Omega(g(x))$ then $f(x) \notin o(g(x))$.*

Proof. A direct application of Lemma 39. □

Lemma 84. *Let g be asymptotically positive. If $f(x) \in o(g(x))$ then $f(x) \notin \Omega(g(x))$.*

Proof. A direct application of Lemma 40. □

Lemma 85. *Let g be asymptotically positive. Then $f(x) \in \Theta(g(x))$ and $f(x) \in \omega(g(x))$ are not both true.*

Proof. A direct application of Lemma 41. □

Lemma 86. *Let g be asymptotically positive. If $f(x) \in \Theta(g(x))$ then $f(x) \notin \omega(g(x))$.*

Proof. A direct application of Lemma 85. □

Lemma 87. *Let g be asymptotically positive. If $f(x) \in \omega(g(x))$ then $f(x) \notin \Theta(g(x))$.*

Proof. A direct application of Lemma 85. □

Lemma 88. *Let g be asymptotically positive. Then $f(x) \in o(g(x))$ and $f(x) \in \omega(g(x))$ are not both true.*

Proof. A direct application of Lemma 44. □

Lemma 89. *Let g be asymptotically positive. If $f(x) \in o(g(x))$ then $f(x) \notin \omega(g(x))$.*

Proof. A direct application of Lemma 88. □

Lemma 90. *Let g be asymptotically positive. If $f(x) \in o(g(x))$ then $f(x) \notin \omega(g(x))$.*

Proof. A direct application of Lemma 88. □

3.3 Reflexivity

Lemma 91. $f(x) \in \Theta(f(x))$.

Proof. Direct consequence of Lemma 47. □

Lemma 92. $f(x) \in O(f(x))$.

Proof. This follows directly from Lemma 48. □

Lemma 93. $f(x) \in \Omega(f(x))$.

Proof. This follows directly from Lemma 49. □

3.4 Transitivity

Lemma 94. *If* $f(x) \in O(g(x))$ *and* $g(x) \in O(h(x))$*, then* $f(x) \in O(h(x))$*.*

Proof. This follows directly from Lemma 50. □

Lemma 95. *If* $f(x) \in \Omega(g(x))$ *and* $g(x) \in \Omega(h(x))$*, then* $f(x) \in \Omega(h(x))$*.*

Proof. This follows directly from Lemma 51. □

Lemma 96. *If* $f(x) \in \Theta(g(x))$ *and* $g(x) \in \Theta(h(x))$*, then* $f(x) \in \Theta(h(x))$*.*

Proof. This follows directly from Lemma 52. □

Lemma 97. *If* $f(x) \in o(g(x))$ *and* $g(x) \in o(h(x))$*, then* $f(x) \in o(h(x))$*.*

Proof. This follows directly from Lemma 53. □

Lemma 98. *If* $f(x) \in \omega(g(x))$ *and* $g(x) \in \omega(h(x))$*, then* $f(x) \in \omega(h(x))$*.*

Proof. This follows directly from Lemma 54. □

3.5 Scalar multiplicativity

Lemma 99. *Let* $c > 0$*. If* $f(x) \in O(g(x))$*, then* $c \cdot f(x) \in O(g(x))$*.*

Proof. This follows directly from Lemma 55. □

Lemma 100. *Let* $c > 0$*. If* $f(x) \in \Omega(g(x))$*, then* $c \cdot f(x) \in \Omega(g(x))$ *is also bounded below by* g*.*

Proof. This follows directly from Lemma 56. □

Lemma 101. *Let* $c > 0$*. If* $f(x) \in \Theta(g(x))$*, then* $c \cdot f(x) \in \Theta(g(x))$*.*

Proof. This follows directly from Lemma 57. □

Lemma 102. *Let* $c < 0$*. If* $f(x) \in O(g(x))$*, then* $c \cdot f(x) \in O(-g(x))$*.*

Proof. This follows directly from Lemma 58. □

Lemma 103. *Let* $c < 0$*. If* $f \in \Omega(g(x))$*, then* $c \cdot f(x) \in \Omega(-g(x))$*.*

Proof. This follows directly from Lemma 59. □

Lemma 104. *Let* $c < 0$*. If* $f(x) \in \Theta(g(x))$*, then* $c \cdot f \in \Theta(-g(x))$*.*

Proof. This follows directly from Lemma 60. □

3.6 Additivity

Lemma 105. *Let* $f\_1(x), f\_2(x) \in O(g(x))$*. Then* $f\_1(x) + f\_2(x) \in O(g(x))$*.*

Proof. This follows directly from Lemma 61. □

Lemma 106. *Let* $f\_1(x), f\_2(x) \in \Omega(g(x))$*. Then* $f\_1(x) + f\_2(x) \in \Omega(g(x))$*.*

Proof. This follows directly from Lemma 62. □

Lemma 107. *Let* $f\_1(x), f\_2(x) \in \Theta(g(x))$*. Then* $f\_1(x) + f\_2(x) \in \Theta(g(x))$*.*

Proof. This follows directly from Lemma 63. □

Lemma 108. *Let* $f\_1(x) \in \Omega(g(x))$ *and let* f_2 *be asymptotically positive. Then* $f\_1(x) + f\_2(x) \in \Omega(g(x))$*.*

Proof. This follows directly from Lemma 64. □

Lemma 109. *Let* $f\_1(x) \in O(g(x))$ *and let* f_2 *be asymptotically negative. Then* $f\_1(x) + f\_2(x) \in O(g(x))$*.*

Proof. This follows directly from Lemma 65. □

Lemma 110. *Let* $f\_1(x) \in \Theta(g(x))$*. Let also* f_2 *be asymptotically positive and* $f\_2(x) \in O(g(x))$*. Then* $f\_1(x) + f\_2(x) \in \Theta(g(x))$*.*

Proof. This follows directly from Lemma 66. □

Lemma 111. *Let* $f\_1(x) \in \Theta(g(x))$*. Let also* f_2 *be asymptotically negative and* $f\_2(x) \in \Omega(g(x))$*. Then* $f\_1(x) + f\_2(x) \in \Theta(g(x))$*.*

Proof. This follows directly from Lemma 67. □

Lemma 112. *Let* $f\_1(x) \in \Theta(g(x))$*. Let also* f_2 *be asymptotically positive and* $f\_2(x) \in o(g(x))$*. Then* $f\_1(x) + f\_2(x) \in \Theta(g(x))$*.*

Proof. This follows directly from Lemma 68. □

Theorem 113. *Let* $f\_1(x) \in \Theta(g(x))$*. Let also* f_2 *be asymptotically negative and* $f\_2(x) \in \omega(g(x))$*. Then* $f\_1(X) + f\_2(x) \in \Theta(g(x))$*.*

Proof. This follows directly from Lemma 69. □

3.7 Multiplicativity

Lemma 114. *Let* f_1 *and* f_2 *be asymptotically nonnegative functions such that* $f\_1(x) \in O(g\_1(x))$ *and* $f\_2(x) \in O(g\_2(x))$*. Then* $f\_1(x) * f\_2(x) \in O(g\_1(x) * g\_2(x))$*.*

Proof. This follows directly from Lemma 70. □

Lemma 115. *Let* f_1 *and* f_2 *be asymptotically nonpositive functions such that* $f\_1(x) \in \Omega(g\_1(x))$ *and* $f\_2(x) \in \Omega(g\_2(x)$*. Then* $f\_1(x) * f\_2(x) \in \Omega(g\_1(x) * g\_2(x))$*.*

Proof. This follows directly from Lemma 71. □

Chapter 4

Geometric sums

Before we finally state and prove the Master theorem, we need to prove two basic properties regarding geometric sums. These are both straightforward proofs with no clever tricks or surprises, but are still stated for the sake of completeness.

Definition 116. Let $x \in \mathbb{R}$ and $n \in \mathbb{N}$. A **geometric sum** is a sum of the form

$$\sum_{k=0}^{n} x^k.$$

Proposition 117. *Let $x \neq 1$ and $n \in \mathbb{N}$. Then the following inequality holds:*

$$\sum_{k=0}^{n} x^k \leq \frac{x^{n+1}-1}{x-1}$$

Proof. We prove the proposition by induction on n. The base case is simple:

$$\begin{aligned}\frac{x^{0+1}-1}{x-1} &= \\ &= \frac{x^1-1}{x-1} \\ &= \frac{x-1}{x-1} \\ &= 1 \\ &= x^0\end{aligned}$$

The inductive step is also straightforward:

$$\begin{aligned}\sum_{k=0}^{n} x^k &= \sum_{k=0}^{n-1} x^k + x^n \\ &= \frac{x^n-1}{x-1} + \frac{x-1}{x-1}x^n \\ &= \frac{x^n-1}{x-1} + \frac{x^{n+1}-x^n}{x-1} \\ &= \frac{x^{n+1}-1}{x-1}\end{aligned}$$

□

Proposition 118. *Let* $0 < x < 1$ *and* $n \in \mathbb{N}$. *Then the following inequality holds:*

$$\sum_{k=0}^{n} x^k \leq \frac{1}{1-x}$$

Proof. The inequality follows directly from the previous proposition and the restriction on values of x:

$$\begin{aligned}
\sum_{k=0}^{n} x^k &= \frac{x^{n+1} - 1}{x - 1} \\
&= \frac{1 - x^{n+1}}{1 - x} \\
&\leq \frac{1 - x^{n+1}}{1 - x}
\end{aligned}$$

□

Chapter 5

The Master theorem

Analyzing the time complexity of algorithms, especially recursive ones, is more often than not a non-trivial task. For a recursive algorithm, its time complexity can be written as a recurrence formula, which is generally not easy, sometimes even impossible to solve with a closed formula. In some cases, though, we can find asymptotic bounds of the solution, despite not being able to necessarily find the precise solution to the recurrence. One large class of such cases is the class of divide-and-conquer algorithms, i.e. algorithms that recursively split the problem into smaller, similarly-sized subproblems. The Master theorem puts asymptotic bounds on divide-and-conquer recurrences.

Theorem 119 (Master theorem for divide-and-conquer recurrences)**.** *Let $T f : \mathbb{N} \to \mathbb{N}$ be functions such that the recurrence*

$$T(n) = aT(n/b) + f(n)$$

holds for some $a > 0$ and $b > 1$. Let also $f(n) \in \Theta(n^d)$ for some $d \geq 1$. Then the following holds:

1. *if $a < b^d$, then $T(n) \in \Theta(n^d)$,*
2. *if $a = b^d$, then $T(n) \in \Theta(n^d \log_b n)$ and*
3. *if $a > b^d$, then $T(n) \in \Theta(n^{\log_b a})$.*

In the case where f is bounded above by n^d only above or only below, T is also bounded only above or only below by the respective function.

We prove this theorem by proving all of its cases separately.

Definition 120. Let $T f : \mathbb{N} \to \mathbb{N}$ be functions such that T is monotone and the recurrence

$$T(n) \leq aT(\lceil n/b \rceil) + f(n)$$

holds for all $n \geq n\_0$ for some $a > 0$, $n\_0 > 1$ and $b > n\_0$. Let also $f(n) \in O(n^d)$ for some $d \geq 1$. The above recurrence is an **upper master recurrence** with parameters (a, n_0, b, d).

Definition 121. Let $T f : \mathbb{N} \to \mathbb{N}$ be functions such that T is monotone and the recurrence

$$T(n) \geq aT(\lfloor n/b \rfloor) + f(n)$$

holds for all $n \geq n\_0$ for some $a > 0$, $n\_0 > 1$ and $b > n\_0$. Let also $f(n) \in \Omega(n^d)$ for some $d \geq 1$. The above recurrence is a **lower master recurrence** with parameters (a, n_0, b, d).

Lemma 122. *Let T and f form a lower master recurrence with parameters (a, n_0, b, d). Then $T(n) \in O(n^{\log_b a} + \sum_{k=0}^{\lfloor \log_b n \rfloor} (\frac{a}{b^d})^k n^d)$.*

Proof. To eliminate ceilings in the recurrence, we substitute $T(n)$ with $S(n) = T(n+b)$. Since T is monotone, $T(n) \leq S(n) = T(n+b)$ holds for all n. Therefore, an upper bound on S is also an upper bound on T. We must first show that S follows the recurrence of T without ceilings. By the assumption that b is a natural number such that $b > 1$, we have the inequality

$$\begin{aligned} \lceil \frac{n+b}{b} \rceil &\leq \frac{n+b}{b} + 1 \\ &= \frac{n}{b} + 2 \\ &\leq \frac{n}{b} + b. \end{aligned}$$

Therefore, by monotonicity of T we have

$$\begin{aligned} S(n) &= T(n+b) \\ &\leq aT(\frac{n}{b} + b) + f(n+b) \\ &= aS(\frac{n}{b}) + f(n+b), \end{aligned}$$

which captures the wanted recurrence. Integer division here is defined as the floor of real-number division. By substituting the inequality into itself repeatedly, we get

$$\begin{aligned} S(n) &\leq aS(\frac{n}{b}) + f(n+b) && (5.1) \\ &\leq aS(\frac{n}{b}) + Cn^d && (5.2) \\ &\leq a^2 S(\frac{n}{b^2}) + Cn^d + C\frac{a}{b^d}n^d && (5.3) \\ &\leq a^3 S(\frac{n}{b^3}) + C(1 + \frac{a}{b^d} + (\frac{a}{b^d})^2)n^d && (5.4) \\ &\leq \ldots && (5.5) \\ &\leq a^k S(\frac{n}{b^k}) + C\sum_{i=0}^{k}(\frac{a}{b^d})^i n^d && (5.6) \\ & && (5.7) \end{aligned}$$

Let N be the integer such that for all $n \geq N \geq n\_0$, the inequality $f(n) \leq Cn^d$ holds. Such N exists because $f(n) \in O(n^d)$. We set $k = \lfloor \log_b \frac{n}{N} \rfloor$. This choice of k allows the inequality $n \geq N * b^k$ to hold for large enough n.

Consider both sum parts on the right side of the Equation 5.1. For the first part, we notice that $n^{\log_b a} = a^{\log_b n}$. Expand k in the exponent and apply monotonicity:

$$\begin{aligned} a^k &= a^{\lfloor \log_b \frac{n}{N} \rfloor} \\ &\leq a^{\log_b \frac{n}{N}} \\ &\leq a^{\log_b n} \end{aligned}$$

This implies $a^k \in O(n^{\log_b a})$. We also need $S(\frac{n}{b^k})$ to be bounded by a constant. We show that $S(\frac{n}{b^k}) \leq S(N * b)$. Since S is monotone, this is equivalent to showing $\frac{n}{b^k} \leq N * b$. We rewrite the left side as

$$\begin{aligned}\frac{n}{b^k} &= \frac{n}{b^{\lfloor \log_b \frac{n}{N} \rfloor}} \\ &= \frac{n}{b^{\lfloor \log_b n - \log_b N \rfloor}} \\ &= \frac{b^{\log_b n}}{b^{\lfloor \log_b n - \log_b N \rfloor}} \\ &= b^{\log_b n - \lfloor \log_b n - \log_b N \rfloor}\end{aligned}$$

Then, rewrite the right side as $N * b = b^{\log_b N + 1}$. With both sides written as exponents of b, we only need to prove

$$\log_b n - \lfloor \log_b n - \log_b N \rfloor \leq \log_b N + 1.$$

By swapping terms, we get

$$\log_b n - \log_b N \leq \lfloor \log_b n - \log_b N \rfloor + 1,$$

which holds for all real numbers.

For the second part, the inequality $k \leq \lfloor \log_b n \rfloor$ holds by monotonicity of logarithms. As geometric sums are monotone in the exponent, we get $\sum_{i=0}^{k} (\frac{a}{b^d})^i n^d \in O(\sum_{i=0}^{\lfloor \log_b n \rfloor} (\frac{a}{b^d})^i n^d)$. □

Theorem 123. *Let T and f form a lower master recurrence with parameters (a, n_0, b, d), where $a < b^d$. Then $T(n) \in O(n^d)$.*

Proof. First, we apply Lemma 122. Since $a < b^d$, we have $\frac{a}{b^d} < 1$. By basic properties of geometric sums, we get

$$\begin{aligned}T(n) &\leq \sum_{i=0}^{\lfloor \log_b n \rfloor} (\frac{a}{b^d})^i n^d \\ &\leq \frac{1}{1 - \frac{a}{b^d}} n^d,\end{aligned}$$

which proves the upper bound. □

Theorem 124. *Let T and f form an upper master recurrence with parameters (a, n_0, b, d), where $a = b^d$. Then $T(n) \in O(n^d \log_b n)$.*

Proof. After applying Lemma 122, we note that $\log_b a = d$ and then the proof boils down to showing that the geometric sum is bounded by $\log_b n$. Since $\frac{a}{b^d} = 1$, the geometric sum equals $\lfloor \log_b n \rfloor$, which is obviously bounded by $\log_b n$. □

Theorem 125. *Let T and f form an upper master recurrence with parameters (a, n_0, b, d), where $a > b^d$. Then $T(n) \in O(n^{\log_b a})$.*

Proof. After applying Lemma 122, the left side of the sum if trivially bounded by $n^{\log_b a}$. We are left with the right summand, which we transform using an inequality involving the geometric sum:

$$
\begin{aligned}
\sum_{k=0}^{\lfloor \log_b n \rfloor} (\frac{a}{b^d})^k n^d &\leq \frac{1}{\frac{a}{b^d} - 1}((\frac{a}{b^d})^{\lfloor \log_b n \rfloor} - 1)n^d \\
&\leq \frac{1}{\frac{a}{b^d} - 1}((\frac{a}{b^d})^{\log_b n} - 1)n^d \\
&= \frac{1}{\frac{a}{b^d} - 1}(b^{\log_b \frac{a}{b^d}})^{\log_b n} n^d \\
&= \frac{1}{\frac{a}{b^d} - 1}(b^{\log_b n})^{\log_b \frac{a}{b^d}} n^d \\
&= \frac{1}{\frac{a}{b^d} - 1} n^{\log_b a - d} n^d \\
&= \frac{1}{\frac{a}{b^d} - 1} \frac{n^{\log_b a}}{n^d} n^d \\
&= \frac{1}{\frac{a}{b^d} - 1} n^{\log_b a}
\end{aligned}
$$

Since $a > b^d$, $\frac{a}{b^d} > 1$ holds, so $\frac{1}{\frac{a}{b^d}-1} > 0$, which proves boundedness by $n^{\log_b a}$. □

Lemma 126. *Let T and f form an upper master recurrence with parameters (a, n_0, b, d). Then $T(n) \in \Omega(\sum_{k=0}^{\lfloor \log_b n \rfloor} (\frac{a}{b^d})^k n^d)$.*

Proof. We consider the recurrence formula with ceilings replaced by floors. If the resulting inequality holds, then so does the master recurrence, so it suffices to prove the lower bound for this inequality.

By substituting the inequality into itself repeatedly, we get

$$
\begin{aligned}
T(n) &\leq aT(\frac{n}{b}) + f(n+b) &(5.8)\\
&\leq aT(\frac{n}{b}) + Cn^d &(5.9)\\
&\leq a^2T(\frac{n}{b^2}) + Cn^d + C\frac{a}{b^d}n^d &(5.10)\\
&\leq a^3T(\frac{n}{b^3}) + C(1 + \frac{a}{b^d} + (\frac{a}{b^d})^2)n^d &(5.11)\\
&\leq ... &(5.12)\\
&\leq a^kT(\frac{n}{b^k}) + C\sum_{i=0}^{k}(\frac{a}{b^d})^i n^d &(5.13)\\
& &(5.14)
\end{aligned}
$$

We set $k = \lfloor \log_b n \rfloor$. This choice of k allows the inequality $n \geq b^k$ to hold for large enough n. Here C is a positive constant such that $f(n) \geq Cn^d$ for all $n \geq n\_0$. Such a constant exists, because $f(n) \in \Omega(n^d)$ implies $f(n) \geq C\_0n^d$ for some $C\_0 > 0$ for all $n \geq N$ for some N. For $n\_0 \leq n \leq N$, The argument is as follows. The set of natural numbers between n_0 and N is finite, so the image of f on this set has a maximal element M. We then have $f(n) \geq \frac{M}{N^d}n^d$.

The second summand in the right side of Equation 5.8 is trivially bounded above by $\sum_{k=0}^{\lfloor \log_b n \rfloor}(\frac{a}{b^d})^k n^d$. This is sufficient to prove the upper bound od $T(n)$ as the left summand is non-negative. □

Lemma 127. *Let T and f form a lower master recurrence with parameters (a, n_0, b, d). Then $T(n) \in \Omega(n^d)$.*

Proof. Since $T(n) \geq f(n)$ for all $n \geq n\_0$, the lower bound follows directly from $f(n) \in \Omega(n^d)$. $\square$

Theorem 128. *Let T and f form a lower master recurrence with parameters (a, n_0, b, d) where $a = b^d$. Then $T(n) \in \Omega(n^d \log_b n)$.*

Proof. By Lemma 126, it suffices to show that $\sum_{i=0}^{\lfloor \log_b n \rfloor} (\frac{a}{b^d})^i \in \Omega(\log_b n)$. Applying equality $a = b^d$, the sum simplifies to

$$\sum_{i=0}^{\lfloor \log_b n \rfloor} (\frac{a}{b^d})^i = \sum_{i=0}^{\lfloor \log_b n \rfloor} 1^i \qquad = \lfloor \log_b n \rfloor.$$

$\square$

Theorem 129. *Let T and f form an upper master recurrence with parameters (a, n_0, b, d) where $a > b^d$. Then $T(n) \in \Omega(n^{\log_b a})$.*

Proof. By Lemma 126, we need to show that $\sum_{i=0}^{\lfloor \log_b n \rfloor} (\frac{a}{b^d})^i n^d \in \Omega(n^{\log_b a})$. By $a > b^d$, we have

$$\begin{aligned}
\sum_{i=0}^{\lfloor \log_b n \rfloor} (\frac{a}{b^d})^i n^d &\geq (ab^{-d} - 1)^{-1}((ab^{-d})^{\lfloor \log_b n \rfloor} - 1)n^d \\
&\geq 2^{-1}(ab^{-d} - 1)^{-1}(ab^{-d})^{\lfloor \log_b n \rfloor} n^d \\
&\geq 2^{-1}(ab^{-d} - 1)^{-1}(ab^{-d})^{\log_b n - 1} n^d \\
&\geq 2^{-1}a^{-1}b^d(ab^{-d} - 1)^{-1}a^{\log_b n}(b^{\log_b n - 1})^{-d} n^d \\
&\geq 2^{-1}a^{-1}b^d(ab^{-d} - 1)^{-1}n^{\log_b a} n^{-d} n^d \\
&\geq 2^{-1}a^{-1}b^d(ab^{-d} - 1)^{-1}n^{\log_b a},
\end{aligned}$$

which proves the bound. $\square$

Corollary 130. *Let T and f form an upper and lower master recurrence with parameters (a, n_0, b, d) where $a < b^d$. Then $T(n) \in \Theta(n^d)$.*

Proof. By Theorem 79, it suffices to show lower and upper bounds for T, which we already proved in Theorem 123 and Lemma 127. $\square$

Corollary 131. *Let T and f form an upper and lower master recurrence with parameters (a, n_0, b, d) where $a = b^d$. Then $T(n) \in \Theta(n^d \log_b a)$.*

Proof. By Theorem 79, it suffices to show lower and upper bounds for T, which we already proved in Theorems 124 and 128. $\square$

Corollary 132. *Let T and f form an upper and lower master recurrence with parameters (a, n_0, b, d) where $a > b^d$. Then $T(n) \in \Theta(n^{\log_b n})$.*

Proof. By Theorem 79, it suffices to show lower and upper bounds for T, which we already proved in Theorems 125 and 129. $\square$

Chapter 6

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